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The Limits of a Chebyshev-Type Theory of Restricted Range Approximation

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The Chebyshev-type theory of restricted range approximation includes existence, alternation, characterization, and uniqueness. In this paper a detailed study is made of the limits of this theory.

I. INTRODUCTION

Consider extended real-valued function $l(x)$, $u(x)$ on $X \equiv [0, 1]$ subject to the following restrictions:

(i) $-\infty \leqslant l(x) < u(x) \leqslant +\infty$ for all $x \in X$;

(ii) l and u are upper and lower semicontinuous on X, respectively.

Let H be an n-dimensional subspace of $C(X)$ with the Chebyshev norm and set $K = \{q \in H\colon\ l \leqslant q \leqslant u\}$. The problem of restricted range approximation is, given $f \in C(X)$, to find a function $p \in K$ such that

$$\|f - p\| = \inf_{q \in K} \|f - q\|.$$

Such a function p is said to be a best approximation to f from K.

Many authors [1, 2] have studied this problem and have obtained a Chebyshev-type theory including existence, characterization, and uniqueness. In what follows we shall briefly describe this theory. To this end we introduce some notation.

Given $f \in C(X)$ and $p \in K$, denote

$$X_{+1} = \{x \in X\colon f(x) - p(x) = \|f - p\|\},$$

$$X_{-1} = \{x \in X\colon f(x) - p(x) = -\|f - p\|\},$$

$$X_{+2} = \{x \in X\colon p(x) = l(x)\},$$

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$$X_{-2} = \{x \in X: p(x) = u(x)\},$$

$$X_+ = X_{+1} \cup X_{+2},$$

$$X_- = X_{-1} \cup X_{-2},$$

$$X_p = X_+ \cup X_-;$$

$$s(x) = \begin{cases} 1, & x \in X_+ \\ -1, & x \in X_- \end{cases} \quad \text{when } X_+ \cap X_- = \varnothing;$$

$f - p$ is said to alternate k times on X if there are $k+1$ points

$$0 \leqslant x_1 < x_2 < \cdots < x_{k+1} \leqslant 1 \tag{1}$$

in X_p such that the x_i's alternately belong to X_+ and X_-. When $X_+ \cap X_- = \varnothing$ this can be expressed as

$$s(x_{i+1}) = -s(x_i), \qquad i = 1, \ldots, k.$$

The main points of the Chebyshev-type theory of restricted range approximation are summarized in the following theorem; here $\tilde{C}(X) = \{f \in C(X): l \leqslant f \leqslant u\}$.

THEOREM A. *Let H be a Haar subspace. Then*

(a) *For every $f \in C(X)$, f possesses a best approximation from K;*

(b) *For every $f \in C(X)$, a necessary and subbicient condition that $p \in K$ be a best approximation of f from K is that either $X_+ \cap X_- \neq \varnothing$ or $f - p$ alternate at least n times on X, especially*

($\tilde{\text{b}}$) *For every $f \in \tilde{C}(X)$, a necessary and sufficient condition that $p \in K$ be a best approximation to f from K is that $f - p$ alternate at least n times on X;*

(c) *For every $f \in C(X)$, if $X_+ \cap X_- = \varnothing$ for a best approximation to f then the best approximation to f from K is unique, especially*

($\tilde{\text{c}}$) *For every $f \in \tilde{C}(X)$, the best approximation to f from K is unique.*

Remark. The restrictions on l and u are somewhat more relaxed than ones in [1]. But it is easy to see that the theory with these constraints is the same as developed in [1].

In [3, p. 80] a detailed study was made of the limits of a Chebyshev-type theory of approximation without constraints. The purpose of this paper is to consider the limits of a Chebyshev-type theory for restricted range approximation.

In order to state our problems precisely we need

DEFINITION 1. K has Property C (resp. Property $\tilde{C}$) if for every function $f \in C(X)$ (resp. $f \in \tilde{C}(X)$), a necessary and sufficient condition that $p \in K$ be a best approximation to f from K is that $f - p$ alternate at least n times on X.

DEFINITION 2. K has Property C^* (resp. Property $\tilde{C}^*$) if for every function $f \in C(X)$ (resp. $f \in \tilde{C}(X)$), a necessary and sufficient condition that $p \in K$ be a best approximation of f from K is that either $X_+ \cap X_- \neq \emptyset$ or $f - p$ alternate at least n times on X.

DEFINITION 3. K has Property U (resp. Property $\tilde{U}$) if for every function $f \in C(X)$ (resp. $f \in \tilde{C}(X)$), the best approximation to f from K is unique.

Our problems are as follows (X & Y means "both X and Y").

1. What conditions on K are necessary and sufficient for K to have each of Property C, Property $\tilde{C}$, Property C^*, and Property $\tilde{C}^*$?

2. What conditions on K are necessary and sufficient for K to have each of Property C & U, Property $\tilde{C}$ & $\tilde{U}$, Property C^* & U, and Property $\tilde{C}^*$ & $\tilde{U}$?

3. What conditions on K are necessary and sufficient for K to have each of Property U and Property $\tilde{U}$?

We will give a complete answer to each of the above problems. The results for Problems 1, 2, and 3 are given in Sections II, III, and IV, respectively. The last section, Section V, is devoted to summarizing all these results.

II. THE LIMITS OF A CHEBYSHEV-TYPE THEORY—CHARACTERIZATION

The following is from [3, p. 71].

DEFINITION 4. K is said to have Property Z (of degree n) if $p_1, p_2 \in K$, $p_1 \neq p_2$, implies that $p_1(x) - p_2(x)$ has at most $n - 1$ zeros in X.

We need two lemmas for establishing the first main result.

LEMMA 1. *If K has Property $\tilde{C}$, then K has Property Z of degree n.*

Proof. The proof is similar to that of Lemma 3–10 in [3] with the modified definition of f such that $f \in \tilde{C}(X)$:

$$f(x)=\begin{cases} & x\in[x_j+\delta, x_{j+1}-\delta], \qquad j=2,3,...,n\\ M(x) & x\in[0, x_1-\delta] \qquad \text{if } x_1>0\\ & x\in[x_{n+1}+\delta, 1] \qquad \text{if } x_{n+1}<1\\ \min\{p_1(x_j)+\tfrac{3}{2}\eta, u(x_j)\}, & x=x_j,\ j \text{ odd}, 1\leqslant j\leqslant n+1\\ \max\{p_1(x_j)-\tfrac{3}{2}\eta, l(x_j)\}, & x=x_j,\ j \text{ even}, 1\leqslant j\leqslant n+1. \end{cases}$$

In the remaining subintervals of $[0, 1]$, $f(x)$ is defined so that $f\in \tilde{C}(X)$ and

$$|p_1(x)-f(x)|<\tfrac{3}{2}\eta, \qquad |p_2(x)-f(x)|<\tfrac{3}{2}\eta.$$

LEMMA 2. *If K has Property Z and K contains an interior point, i.e., a point p satisfying $l<p<u$, then H is a Haar subspace.*

Proof. It is easy to see that if K contains an interior point then K must contain an n-dimensional neighborhood of this point. Thus by Lemma 3–14 in [3] it follows that H is a Haar subspace.

In order to state our main theorem we need the following

DEFINITION 5. $p\in K$ is said to alternate k times with respect to (l, u) on X if there are $k+1$ points (1) in X such that

$$\text{either } p(x_j)=\begin{cases} l(x_j), & j \text{ odd}\\ u(x_j), & j \text{ even}\end{cases} \qquad \text{or } p(x_j)=\begin{cases} l(x_j), & j \text{ even}\\ u(x_j), & j \text{ odd}.\end{cases} \tag{2}$$

DEFINITION 6. K is said to be an alternation singleton if K contains only one element and this element alternates at least n times with respect to (l, u) on X.

The first main result is

THEOREM 1. *K has Property $\tilde{C}$ if and only if either K is an alternation singleton or H is a Haar subspace.*

Proof. The "if" portion of this theorem follows directly from Theorem A($\tilde{b}$). We proceed with the "only if" portion.

Let $p\in K$. There are two cases to be discussed.

Case 1. p alternates at least n times with respect to (l, u). Then K is a singleton and is, indeed, an alternation singleton. In fact, assume $q\in K$, $q\neq p$. Then there exist $n+1$ points

$$0\leqslant x_1<x_2<\cdots<x_{n+1}\leqslant 1$$

such that either

$$(-1)^j(p(x_j)-q(x_j)) \geqslant 0, \qquad j=1, 2, ..., n+1$$

or

$$(-1)^{j+1}(p(x_j)-q(x_j)) \geqslant 0, \qquad j=1, 2, ..., n+1,$$

each of which implies that $p-q$ has at least n zeros by the Assertion in [3, p. 61]. This is a contradiction, because by Lemma 1, K has Property Z.

Case 2. p alternates exactly k times with respect to (l, u), $k<n$. By Lemma 1 and Lemma 2 it suffices to show that K contains an interior point.

The interval [0, 1] may be divided into $k+1$ subintervals by

$$0=x_0<x_1< \cdots <x_k<x_{k+1}=1$$

so that either

$$p(x)>l(x)$$

or

$$p(x)<u(x)$$

is alternately valid on the subinterval $[x_j, x_{j+1}]$, $j=0, 1, ..., k$. For concreteness, assume that

$$\begin{aligned} &p(x)>l(x), \qquad x\in[x_j, x_{j+1}],\ j \text{ even},\ 0\leqslant j\leqslant k \\ &p(x)<u(x), \qquad x\in[x_j, x_{j+1}],\ j \text{ odd},\ 0\leqslant j\leqslant k. \end{aligned} \tag{3}$$

Take $t>0$ small enough so that

$$e_0=\min\{p(x)-l(x), u(x)-p(x): x\in Y_j\equiv[x_j-t, x_j+t], \\ j=1, ..., k\}>0.$$

Denote $X_0=[x_0, x_1-t]$, $X_j=[x_j+t, x_{j+1}-t]$, $j=1, ..., k-1$, $X_k=[x_k+t, x_{k+1}]$. Let

$$e_1=\min\{p(x)-l(x): x\in X_j, j \text{ even}, 0\leqslant j\leqslant k\},$$
$$e_2=\min\{u(x)-p(x): x\in X_j, j \text{ odd}, 0\leqslant j\leqslant k\},$$

and

$$e=\tfrac{1}{2}\min\{e_0, e_1, e_2\}.$$

A function $f \in C(X)$ is defined as

$$f(x) = p(x) - (-1)^j e, \qquad x \in X_j, \quad j = 0, 1, ..., k. \tag{4}$$

In each subinterval Y_j, $j = 1, 2, ..., k$, $f(x)$ intersects $p(x)$ only once and satisfies

$$|p(x) - f(x)| \leqslant e. \tag{5}$$

Let us examine that $f \in \tilde{C}(X)$. For $x \in Y_j$, $j = 1, 2, ..., k$, it follows from (5) that

$$f(x) \leqslant p(x) + e \leqslant p(x) + e_0 \leqslant p(x) + u(x) - p(x) = u(x)$$

and

$$f(x) \geqslant p(x) - e \geqslant p(x) - e_0 \geqslant p(x) + l(x) - p(x) = l(x).$$

And for $x \in X_j$, j even,

$$f(x) = p(x) - e \leqslant p(x) \leqslant u(x)$$

and

$$f(x) = p(x) - e \geqslant p(x) - e_1 \geqslant p(x) + l(x) - p(x) = l(x).$$

The same conclusion is valid for $x \in X_j$, j odd.

Since $f - p$ alternates exactly k times, $k < n$, by the assumption of the theorem, p is not a best aproximation to f from K. Hence there is a function $p^* \in K$ such that p^* is a better approximation to f than p, i.e.,

$$\|f - p^*\| < \|f - p\| = e. \tag{6}$$

We claim that

$$l < p^* < u. \tag{7}$$

In fact, it follows from (6) that $f - e < p^* < f + e$. Hence for $x \in X_j$, j even, we have $f(x) = p(x) - e$. Whence

$$p^*(x) < f(x) + e = p(x) \leqslant u(x)$$

and

$$p^*(x) > f(x) - e = p(x) - 2e \geqslant p(x) - e_1$$
$$\geqslant p(x) - (p(x) - l(x)) = l(x).$$

A similar argument is applicable to $x \in X_j$, j odd.

Further, for $x \in Y_j$, $j = 1, \ldots, k$, we obtain that

$$p^*(x) < f(x) + e \leqslant p(x) + 2e \leqslant p(x) + e_0 \leqslant p(x) + u(x) - p(x) = u(x)$$

and

$$p^*(x) > f(x) - e \geqslant p(x) - 2e \geqslant p(x) - e_0 \geqslant p(x) - (p(x) - l(x)) = l(x).$$

This proves that $l < p^* < u$ and concludes the proof of the theorem.

DEFINITION 7. K is said to have Property B_1 if $n > 1$ implies $K = H$, and $n = 1$ and $l(x) \not\equiv -\infty$ (resp. $u(x) \not\equiv +\infty$) imply the existence of two distinct points x_1, x_2 and a function $p_0 \in K$ satisfying

$$p_0(x_i) = l(x_i) \qquad (\text{resp. } u(x_i)), \qquad i = 1, 2. \tag{8}$$

The next main result is

THEOREM 2. *K has Property C if and only if either K is an alternation singleton or H is a Haar subspace and K has Property B_1.*

Proof. Sufficiency. It is easy to see that if K is an alternation singleton or $K = H$ is a Haar subspace, then K has Property C. So we need only to show that if H is a Haar subspace and K satisfies Property B_1 for $n = 1$, then K has Property C.

If $X_+ \cap X_- = \varnothing$, by Theorem A, Property C is valid.

On the other hand, $X_+ \cap X_- \neq \varnothing$ means that one of the following three cases occurs.

Case 1. $X_{+1} \cap X_{-1} \neq \varnothing$. This means $f \in K$. A set of any two distinct points provides an alternation once.

Case 2. $X_{-1} \cap X_{+2} \neq \varnothing$. That implies $l(x) \not\equiv -\infty$. Let $x_0 \in X_{-1} \cap X_{+2}$. Then

$$p(x_0) - f(x_0) = \|f - p\| \qquad \text{and} \qquad p(x_0) = l(x_0), \tag{9}$$

where $p \in K$ is a best approximation to f. Since H is a Haar subspace, it follows from $p(x_0) = l(x_0)$ and (8) that $p = p_0$. Thus $f - p$ alternates at least once, since $x_0 \in X_+$ and $x_1, x_2 \in X_-$.

Case 3. $X_{+1} \cap X_{-2} \neq \varnothing$. A similar argument is valid for this case.

Necessity. Assume that K is not an alternation singleton. By Theorem 1, H is a Haar subspace.

Let $n = 1$. The proof is given for $l(w) \not\equiv -\infty$. A similar proof is valid for $u(x) \not\equiv +\infty$. Since $l(x) \not\equiv -\infty$ and l is upper semicontinuous, there is a function $p_0 \in K$ such that $p_0(x_1) = l(x_1)$ for some x_1. Since K is not an alternation singleton, $p_0(x) < u(x)$ for all $x \in X$. Suppose on the contrary that $p_0(x) \neq l(x)$ for any $x \neq x_1$. Then the function

$$f(x) = p_0(x) + |x - x_1| - 1, \qquad x \in X \tag{10}$$

has the properties

$$|f(x_1) - p_0(x_1)| = \|f - p_0\| \quad \text{and} \quad |f(x) - p_0(x)| < \|f - p_0\|, \quad x \neq x_1. \tag{11}$$

Since $x_1 \in X_{-1} \cap X_{+2}$, by Theorem A, p_0 is a best approximation to f from K. On the other hand, $f - p_0$ has no alternation, a contradiction. This concludes Property B_1 for $n = 1$.

Let $n > 1$. Suppose on the contrary that $l(x) \not\equiv -\infty$. By the proof of Theorem 1 there is a function $p^* \in K$ such that $p^*(x) < u(x)$ for all $x \in X$. Then there exists a function $p_0 \in K$ such that

$$\begin{aligned} p_0(x) &< u(x) && \text{for all } x \in X \\ p_0(x_1) &= l(x_1) && \text{for some } x_1 \in X \end{aligned}$$

In fact, p_0 is a solution of the minimization problem

$$\inf_{x \in X} (p_0(x) - l(x)) = \inf_{q \in K_1} \inf_{x \in X} (q(x) - l(x))$$

in $K_1 = \{q \in K \colon l \leqslant q \leqslant p^*\}$ and always exists. Without loss of generality assume that x_1 satisfies that $p_0(x) \neq l(x)$ for all $x < x_1$, otherwise we replace x_1 by a point satisfying this condition. Then the function f defined by (10) satisfies (11) and has p_0 as a best approximation. But $f - p_0$ alternates at most once, which contradicts Property C and $n > 1$. This proves $l = -\infty$. Similarly, we must have $u = +\infty$.

THEOREM 3. *K has Property $\tilde{C}^*$ if and only if either K is an alternation singleton or H is a Haar subspace.*

Proof. It suffices to show that Property $\tilde{C}^*$ is equivalent to Property $\tilde{C}$. Clearly, Property $\tilde{C}$ implies Property $\tilde{C}^*$. On the other hand, since $u > l$ implies that either $f \in K$ or $X_+ \cap X_- = \varnothing$, Property $\tilde{C}^*$ implies Property $\tilde{C}$.

THEOREM 4. *K has Property C^* if and only if either K is an alternation singlton or H is a Haar subspace.*

Proof. Since Property C^* implies Property $\tilde{C}^*$, by Theorem 3, Property

C^* implies Property $\tilde{C}$. Conversely, the "if" portion of the theorem follows directly from Theorem A.

III. THE LIMITS OF A CHEBYSHEV-TYPE THEORY—CHARACTERIZATION AND UNIQUENESS

If follows immediately from Theorems 1 and 2 that

LEMMA 3. (a) *Property* $\tilde{C}$ *implies Property* $\tilde{U}$.

(b) *Property* C *implies Property* U.

By Theorems 1, 2, and 3 and Lemma 3 we can easily obtain the following three theorems.

THEOREM 5. *K has Property $\tilde{C}\&\tilde{U}$ if and only if either K is an alternation singleton or H is a Haar subspace.*

THEOREM 6. *K has Property $C\&U$ if and only if either K is an alternation singleton or H is a Haar subspace and K has Property B_1.*

THEOREM 7. *K has Property $\tilde{C}\&\tilde{U}$ if and only if either K is an alternation singleton or H is a Haar subspace.*

The next theorem, which characterizes Property $C^*\&U$, is somewhat difficult. We begin with

DEFINITION 8. K has Property B_2 if either $l = -\infty$ (resp. $u = +\infty$) or $p_1, p_2 \in K$ and $p_1(x_0) = p_2(x_0) = l(x_0)$ (resp. $u(x_0)$) imply $p_1 = p_2$.

EXAMPLE. If $l = (\frac{1}{4} - (x - \frac{1}{2})^2)^{1/2}$, $u = +\infty$ and $H = \text{span}\{1, x\}$, it is easy to see that K satisfies Property B_2.

LEMMA 4. *Property U implies Property B_2.*

Proof. If possible, suppose that K does not have Property B_2, and say that $p_1(x_0) = p_2(x_0) = l(x_0)$ for some $p_1, p_2 \in K$, $p_1 \neq p_2$, and $x_0 \in X$. Denote $d = \|p_1 - p_2\|$ and define $f \in C(X)$ by

$$f(x_0) = l(x_0) - d,$$
$$|f(x) - p_i(x)| < d, \qquad x \neq x_0, \quad i = 1, 2.$$

Obviously both p_1 and p_2 are best approximations to f from K by definition. But this contradicts Property U. The lemma is established.

THEOREM 8. *K has Property C*&U if and only if either K is an alternation singleton or H is a Haar subspace and K has Property B_2.*

Proof. ($\Leftarrow$) if K is an alternation singleton, clearly the conclusion is right. If H is a Haar subspaces, by Theorem A, K has Property C^* and the uniqueness is valid for those $f \in C(X)$ for which $X_+ \cap X_- = \varnothing$. Assume that $f \in C(X)$ has a best approximation p and $X_+ \cap X_- \neq \varnothing$, say $x_0 \in X_+ \cap X_-$. Then one of the following three cases occurs:

(1) $x_0 \in X_{+1} \cap X_{-1}$; (2) $x_0 \in X_{-1} \cap X_{+2}$; (3) $x_0 \in X_{+1} \cap X_{-2}$.

Case (1) means $f \in K$ and is trivial. Case (2) means (9). Thus if $p^* \in K$ is also a best approximation to f, then

$$p^*(x_0) - f(x_0) \leqslant \|f - p\|$$

and

$$p^*(x_0) \geqslant l(x_0).$$

By (9) we obtain $p(x_0) = p^*(x_0) = l(x_0)$. By virtue of Property B_2 we conclude $p = p^*$, which shows that the best approximation p to f is unique. A similar argument may establish the uniqueness of best approximation for case (3).

($\Rightarrow$) If K is not an alternation singleton, by Theorem 4, H is a Haar subspace. Also, by Lemma 4, K has Property B_2.

IV. THE LIMITS OF A CHEBYSHEV-TYPE THEORY—UNIQUENESS

For preparation for the proof of Theorem 9, we establish

LEMMA 5. *If K is not a singleton, then Property B_2 and Property Z imply that H is a Haar subspace.*

Proof. Let p_1 and p_2 be in K and $p_1 \neq p_2$. Then $p = \frac{1}{2}(p_1 + p_2)$ must satisfy that $l(x) < p(x) < u(x)$ for all $x \in X$, because $p(x_0) = l(x_0)$ and $p(x_0) = u(x_0)$ lead to $p_1(x_0) = p_2(x_0) = l(x_0)$ and $p_1(x_0) = p_2(x_0) = u(x_0)$, respectively, contradicting Property B_2. If it is coupled with Property Z, then by Lemma 2 we assert that H is a Haar subspace.

The first main theorem in this section is as follows.

THEOREM 9. *K has Property U if and only if either K is a singleton or H is a Haar subspace and K has Property B_2.*

Proof. The "if" portion of the theorem follows directly from Theorem 8. We proceed with the "only if" portion.

By Lemma 3–13 in [3, p. 87], Property U implies Property Z. Lemma 4 says that Property U implies Property B_2. Thus by Lemma 5, H is a Haar subspace if K is not a singleton.

The last main theorem in this section is concerned with the equivalent condition of Propety $\tilde{U}$. For this theorem we need to establish a lemma, which is of independent interest.

LEMMA 6. *Property $\tilde{U}$ implies Property Z.*

Proof. Suppose on the contrary that there are $p_1, p_2 \in K$ with $p_1 \neq p_2$ such that $p_1 - p_2$ has n distinct zeros, say $x_1, x_2, ..., x_n$. Without loss of generality we assume that $e \equiv \frac{1}{2}\|p_1 - p_2\| < \frac{1}{4}\inf\{u(x) - l(x)\colon\ x \in X\}$, otherwise we replace p_1 and p_2 by $(1-t_1)p_1 + t_1 p_2$ and $(1-t_2)p_1 + t_2 p_2$, respectively, with $t_1 \neq t_2$ and $|t_1 - t_2|$ small enough. Define $g \in C(X)$ such that

$$l + e \leqslant g \leqslant u - e \tag{12}$$

and

$$\|g - p_1\| = \|g - p_2\| = e. \tag{13}$$

Such a function g must exist. In fact, (13) is equivalent to

$$p_1 - e \leqslant g \leqslant p_1 + e \qquad \text{and} \qquad p_2 - e \leqslant g \leqslant p_2 + e.$$

Thus it suffices to define $g \in C(X)$ satisfying $l^* \leqslant g \leqslant u^*$, where $l^* = \max\{p_1 - e,\ p_2 - e,\ l + e\}$ and $u^* = \min\{p_1 + e,\ p_2 + e,\ u - e\}$. But $l^* \leqslant u^*$ is, indeed, valid, because if follows from

$$\begin{gathered} \max\{p_1, p_2\} - e \leqslant \min\{p_1, p_2\} + e, \qquad \max\{p_1, p_2\} - e \leqslant u - e, \\ l + e \leqslant u - e \qquad \text{and} \qquad l + e \leqslant \min\{p_1, p_2\} + e. \end{gathered} \tag{14}$$

Since l^* and u^* are upper and lower semicontinuous, respectively, there exists a function $g \in C(X)$ such that $l^* \leqslant g \leqslant u^*$. By uniqueness it follows from (13) that neither p_1 nor p_2 is a best approximation to g from K. Therefore there must exist a function p in K such that p is a better approximation to g than p_1, i.e., $\|g - p\| < \|g - p_1\| = e$. According to (12) we get that $l < p < u$. Write $p_i^* = \frac{1}{2}(p + p_i)$, $i = 1, 2$. Whence

$$l < p_i^* < u, \qquad i = 1, 2 \tag{15}$$

and $p_1^* - p_2^* = \frac{1}{2}(p_1 - p_2)$ also has n zeros $x_1, x_2, ..., x_n$. Without loss of generality we further assume that

$$M \equiv \| p_1^* - p_2^* \| < \min_{i=1,2} \inf_{x \in X} \{u(x) - p_i^*(x), p_i^*(x) - l(x)\} \tag{16}$$

using the techniques as before.

Now we can use a similar argument, as used in the proof of Lemma 3–13 in [3] by Rice, where $f(x)$ may be chosen so that $f \in \tilde{C}(X)$ because of (15) and (16). Therefore both p_1^* and p_2^* are best approximations to f from K. This contradiction proves Property Z.

THEOREM 10. *K has Property $\tilde{U}$ if and only if either K is a singleton or H is a Haar subspace.*

Proof. The "if" portion of the theorem is given by Theorem A. For the "only if" portion we see from Lemma 6 that K has Property Z. If K is not a singleton, Property Z as well as (15) implies that H is a Haar subspace by Lemma 2.

V. SUMMARY

Theorems 1, 3, 4, 5, and 7 can be restated as follows.

THEOREM 11. *The following statements are equivalent to each other*:

(a) *Either K is an alternation singleton or H is a Haar subspace*;

(b) *K has Property $\tilde{C}$*;

(c) *K has Property $\tilde{C}^*$*;

(d) *K has Property C^**;

(e) *K has Property $\tilde{C}\&\tilde{U}$*;

(f) *K has Property $\tilde{C}^*\&\tilde{U}$.*

Theorems 2 and 6 can be restated as follows.

THEOREM 12. *The following statement are equivalent to each other*:

(a) *Either K is an alternation singleton or H is a Haar subspace and K has Property B_1*;

(b) *K has Property C*;

(c) *K has Property $C\&U$.*

From Theorems 1, 3, 4, 5, 7, and 10 we have

THEOREM 13. *If K is not a singleton, then the following statements are equivalent to each other*:

(a) *H is a Haar subspace*;

(b) *K has Property $\tilde{C}$*;

(c) *K has Property $\tilde{C}^*$*;

(d) *K has Property C^**;

(e) *K has Property $\tilde{U}$*;

(f) *K has Property $\tilde{C}\&\tilde{U}$*;

(g) *K has Property $\tilde{C}^*\&\tilde{U}$.*

From Theorems 8 and 9 we have

THEOREM 14. *If K is not a singleton, then the following statements are equivalent to each other*:

(a) *H is a Haar subspace and K has Property B_2*;

(b) *K has Property U*;

(c) *K has Property $C^*\&U$.*

REFERENCES

1. G. D. TAYLOR, Approximation by functions having restricted ranges, III, *J. Math. Anal. Appl.* **27** (1969), 241–248.
2. B. L. CHALMERS AND G. D. TAYLOR, Uniform approximation with constraints, *Jahresber. Deutsch. Math.-Verein.* **81** (1979), 49–86.
3. J. R. RICE, "The Approximation of Functions," Vol. 1, Addison–Wesley, Reading, MA/Palo Alto, CA/London, 1964.